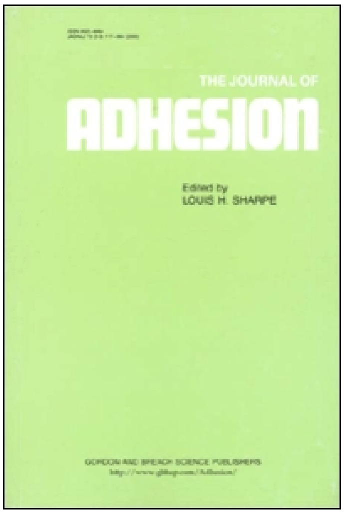

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An Investigation of Non-Linear Viscoelastic Effects on Load Transfer in a Symmetric Double-Lap Joint

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The purpose of this investigation is to explore some of the effects of the non-linear viscoelastic behavior of the adhesive material on the response of a symmetric lap-joint. Although full accounting for the non-linear behavior requires the employment of finite-element methods the semi-analytical solution presented in this work contains a paramount feature of the non-linear character and therefore provides insight into the complex response of the joint.

INTRODUCTION

Analytical and numerical investigations of adhesive joints have shown that the stress field within the adhesive layer is highly non-uniform, with sharp peaks near the ends of the bond line.[1–4] In fact, it is to be expected that a linear-elastic analysis should yield infinite values for the shear and normal stresses at those ends, due to the corner singularity that exists at the point where the adhesive-adherend interface intersects the free edge.[5,6]

These excessively high stresses do not exist in reality. In several analyses these sharp peaks were reduced to the level of the experimentally measured ultimate stress by assuming elasto-plastic or visco-plastic behavior of the adhesive material.[7,8] However, various data indicate that the behavior of the adhesive is characterized by a non-linear viscoelastic relation rather than by plastic yield and flow.[9,10,11]

The non-linear viscoelastic behavior, at which strains are no longer proportional to stresses, is typified by an accelerated and stress-enhanced creep. Basically, at elevated stresses the material moduli seem to soften and the creep to progress at accelerated rates. It is this accelerated creep-rate which is considered in the present work.

VARIATIONAL FORMULATION

Consider the symmetric double-lap joint of length $2c$ shown in Figure 1. Let the thickness of the central adherend and the outer laps be $2h$ and h respectively, and let a be the thickness of the adhesive layer. We consider all adherends to be made of the same material.

The joint is pulled apart by a force P. We designate the center adherend by the letter "C", the outer-adherends by "B" and the adhesive layers by "A."

Due to the symmetry of our problem we place the origin of the x–z axes at the center of the adherend "C" and at a distance $c/2$ from the edge and analyse the quarter portion shown in Figure 2.

Assume that the adherends respond in pure tension and the adhesive in pure shear. A system of field equations and boundary conditions which is consistent with those assumptions and applicable to any constitutive relation is derivable by means of the principle of virtual work.

In view of the above mentioned assumptions and the symmetry of the joint we have the following expression for the variation of the internal energy U:

$$(1/4)\delta U = \int_{-c/2}^{c/2} \left\{ \int_0^h \sigma_x^C \delta \varepsilon_x^C \, dz + \int_h^{h+a} \tau_{xz}^A \delta \gamma_{xz}^A \, dz + \int_{h+a}^{2h+a} \sigma_x^B \delta \varepsilon_x^B \, dz \right\} dx \tag{1}$$

None of the integrands in (1) depends on z, therefore integration across the thickness yields

$$(1/4)\delta U = \int_{-c/2}^{c/2} (N_C \delta \varepsilon_x^C + Q_A \delta \gamma_{xz}^A + N_B \delta \varepsilon_x^B) \, dx \tag{2}$$

where Q_A is the resultant of the shear stress in the adhesive layer and N_B, N_C are the normal stress resultants.

In terms of the displacement u_C and u_B, of the central and outer adherends, we have

$$\begin{aligned} \varepsilon_x^C &= u_C' \\ \varepsilon_x^B &= u_B' \end{aligned} \tag{3}$$

and

$$\gamma_{xz} = \frac{u_B - u_C}{a} \tag{4}$$

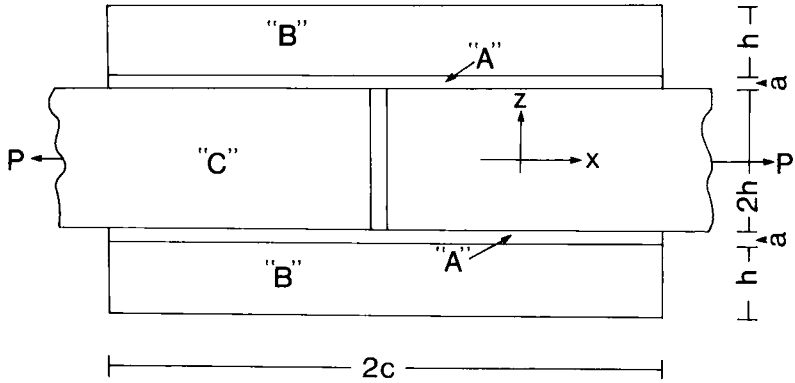

FIGURE 1 The geometry of the adhesive joint.

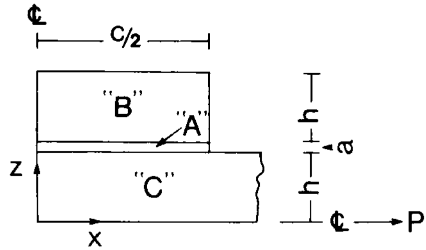

FIGURE 2 The analysed portion of the joint and the prevailing symmetries.

where primes denote derivatives with respect to x. Therefore (2) reads

$$(1/4)\delta U = \int_{-c/2}^{c/2}\left[N_C\delta u_C + N_B\delta u_B + Q_A\left(\frac{\delta u_B - \delta u_C}{a}\right)\right]dx \tag{5}$$

Integration by parts yields

$$(1/4)\delta U = \int_{-c/2}^{c/2}\left[-\left(N_C' + \frac{Q_A}{a}\right)\delta u_C + \left(-N_B' + \frac{Q_A}{a}\right)\delta u_B\right]dx + \left[N_C\delta u_C + N_B\delta u_B\right]_{x=-c/2}^{x=c/2} \tag{6}$$

The variation of the external work W, commensurate with the above form of δU, is given by

$$(1/4)\delta W = \left[N_C^*\delta u_C + N_B^*\delta u_B\right]_{x=-c/2}^{x=c/2} \tag{7}$$

where N_C^*, N_B^* refer to applied external loads.

Employing the principle of virtual work $\delta U - \delta W =$ we obtain the following field equations

$$N'_{\mathrm{C}} + \frac{Q_{\mathrm{A}}}{a} = 0$$
$$N'_{\mathrm{B}} - \frac{Q_{\mathrm{A}}}{\mathrm{a}} = 0 \tag{8}$$

In addition, we also have

$$[(N_{\mathrm{C}} - N^*_{\mathrm{C}})\delta u_{\mathrm{C}} + (N_{\mathrm{B}} - N^*_{\mathrm{B}})\delta u_{\mathrm{B}}]_{x=-c/2}^{x=c/2} = 0 \tag{9}$$

Eq. 9 determines the boundary conditions which are consistent with the present variational formulation. Accordingly, we must prescribe either N_{C} or u_{C}, as well as either N_{B} or u_{B}, at $x = \pm c/2$.

A variant to the field equations 8 and the above mentioned boundary conditions is obtainable by noting that Eq. 8 yield $N'_{\mathrm{C}} + N'_{\mathrm{B}} = 0$. Integrating this last equation we obtain from considerations of global equilibrium

$$N_{\mathrm{C}} + N_{\mathrm{B}} = \frac{P}{2} \tag{10}$$

The field equation 10 can be used in place of any one of Eq. 8. Furthermore, when Eq. 10 is employed at the boundaries $x = \pm c/2$, it can be used to reduce the number of boundary conditions required for a traction problem.

THE LINEAR-ELASTIC SOLUTION

Let E and G denote the Young's modulus and the shear modulus of the adherend and the adhesive, respectively. Then

$$\sigma_x^{\mathrm{C}} = Eu'_{\mathrm{C}}$$
$$\sigma_x^{\mathrm{B}} = Eu'_{\mathrm{B}}$$

and (11)

$$\tau = \tau_{xz} = G\frac{u_{\mathrm{B}} - u_{\mathrm{C}}}{a}$$

whereby

$$N_{\mathrm{C}} = Ehu'_{\mathrm{C}}$$
$$N_{\mathrm{B}} = Ehu'_{\mathrm{B}} \tag{12}$$
$$Q_{\mathrm{A}} = G(u_{\mathrm{B}} - u_{\mathrm{C}})$$

Straightforward manipulations, which are omitted for the sake of brevity, yield the well known solution:[3]

$$\tau = -\frac{P}{4L}\frac{\cosh(x/L)}{\sinh(c/2L)} \tag{13}$$

where

$$L^2 = \frac{ah}{2}\frac{E}{G}$$

Note that as $L \to \infty$ τ tends to the uniform distribution $\tau = -P/2c$. As L diminishes the distribution reaches sharper and sharper peaks at the edges $x = \pm c/2$.

In a similar fashion we can also compute the remaining stresses and displacements in the joint. For instance, we obtain

$$u_C(c/2) = \frac{aP}{4GL}(\gamma + \coth\gamma) = u_C(-c/2) + \frac{aP\gamma}{4GL} \tag{14}$$

where

$$\gamma = c/2L$$

THE NON-LINEAR VISCOELASTIC CASE

Consider now an adhesive material characterized by a non-linear viscoelastic response. Since the adherends are still linearly elastic Eqs. 11_1 and 11_2 remain valid and all the inelasticity resides in 11_3. The time dependent response of many adhesive materials can be expressed by a "power-law" compliance

$$D(t) = D_0 + D_1 t^n \tag{15}$$

where D_0 is the instantaneous compliance, t is time and D_1, n are material constants.

We shall represent the non-linearity of the viscoelastic response by considering a stress enhanced creep.[9–11] To do this we shall introduce a "reduced time" ξ into the time-dependent portion of the shear strain in the adhesive layer, where the time reduction is accomplished by a stress-dependent shift-factor. We thus have $\xi = t/a_\sigma$ where $a_\sigma = \exp(-\theta\tau)$, θ being a material constant.[10]

A complete treatment of the non-linear problem at hand requires the employment of superposition integrals which would entail numerical iterations that are best handled by means of a finite element formulation. To avoid this task we shall use the approximate quasi-elastic method[12] which is known to yield satisfactory results in many realistic applications.

Nevertheless, the validity of the quasi-elastic approximation in the present, non-linear, case must wait verification by comparison with finite element solutions.

Consequently, Eq. 11_3 is replaced by the following expression

$$\frac{u_B - u_C}{a} = \tau[D_0 + D_1(t/e^{-\theta\tau})^n] \tag{16}$$

Denote $\alpha = n\theta$ and $D_1 t^n = q$ then (16) reads

$$\frac{u_B - u_C}{a} = R(\tau) \tag{17}$$

with

$$R(\tau) = \tau(D_0 + qe^{\alpha\tau}) \tag{18}$$

Eqs. 12_1 and 12_2, which remain valid, now yield

$$\frac{N_B - N_C}{Eah} = \frac{dR(\tau)}{dx} = R'(\tau)\tau'(x) \tag{19}$$

Eqs. 8 and 19 give

$$\frac{2Q_A}{Ea^2h} = \frac{d^2R(\tau)}{dx^2}$$

whereby

$$\frac{2\tau}{Eah} = R''(\tau)[\tau'(x)]^2 + R'(\tau)\tau''(x) \tag{20}$$

In (19, (20), and the sequel, primes denote derivatives with respect to the argument.

With the aid of field equations (10), (19) and $(12)_1$, $(12)_2$ we have

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u'_B \\ u'_C \end{bmatrix} = \begin{bmatrix} aR'(\tau)\tau'(x) \\ P/2hE \end{bmatrix} \tag{21}$$

consequently

$$\begin{aligned} u'_B &= \frac{1}{2}\left[\frac{P}{2hE} + aR'(\tau)\tau'(x)\right] \\ u'_C &= \frac{1}{2}\left[\frac{P}{2hE} - aR'(\tau)\tau'(x)\right] \end{aligned} \tag{22}$$

Together with the boundary conditions $N_C(c/2) = N_C(-c/2) = 0$ we get

$$\tau'(c/2) = -\tau'(-c/2) = \frac{P}{2ahE}\frac{1}{R'(\tau(c/2))} \tag{23}$$

AN ITERATIVE SOLUTION

Denoting $Eha/2 = \beta$ we can rewrite (20) as follows

$$\tau''(x) = \frac{1}{R'(\tau)}\left\{\frac{\tau(x)}{\beta} - R''(\tau)[\tau'(x)]^2\right\} \tag{24}$$

In the sequel we shall omit the arguments, with the understanding that $\tau = \tau(x)$, $\tau' = d\tau/dx$, etc. and $R = R(\tau)$, $R' = dR/d\tau$, etc.

Successive differentiation of (24) yields

$$\tau''' = \frac{1}{R'}\left[\frac{\tau'}{\beta} - R'''(\tau')^3 - 3R''\tau'\tau''\right]$$

$$\tau^{(\mathrm{iv})} = \frac{1}{R'}\left\{\frac{\tau''}{\beta} - R^{(\mathrm{iv})}(\tau')^4 - 6R'''(\tau')^2\tau'' - R''[3(\tau'')^2 + 2\tau'\tau''']\right\}$$

$$\tau^{(\mathrm{v})} = \frac{1}{R'}\left\{\frac{\tau'''}{\beta} - R^{(\mathrm{v})}(\tau')^5 - 10R^{(\mathrm{iv})}(\tau')^3\tau'' - R'''[15\tau'(\tau'')^2 + 8(\tau')^2\tau'''] - R''(3\tau'\tau^{(\mathrm{iv})} + 8\tau''\tau''')\right\}$$

where, in (25)

$$R' = r + qe^{\alpha\tau}(1 + \alpha\tau)$$
$$R^{(j)} = q\alpha^{(j-1)}e^{\alpha\tau}(j + \alpha\tau) \qquad j \geqslant 2$$

The iterative scheme employs a guess value for $\tau(0)$, say $\tau(0) = \tau_0^{(1)}$. Due to the symmetry of the problem we have $\tau'(0) = 0$.

We now divide the interval $0 \leqslant x \leqslant c/2$ into N equal sub-intervals* $\Delta = c/2N$ and denote $x_i = i\Delta (i = 1, 2, \ldots N)$ and $\tau(x_i) = \tau_i$.

Taylor expansions, truncated after five terms (to provide sufficient accuracy with a reasonably small N), yield

$$\tau_i = \tau_{i-1} + \Delta\tau'_{i-1} + \cdots + \frac{1}{4!}\Delta^4\tau_{i-1}^{(\mathrm{iv})}:$$
$$\tau'_i = \tau'_{i-1} + \Delta\tau''_{i-1} + \cdots + \frac{1}{4!}\Delta^4\tau_{i-1}^{(\mathrm{v})} \tag{26}$$

Now, upon selecting a guess $\tau(0) = \tau_0^{(1)}$—and with $\tau'(0) = 0$—we can compute $\tau''(0)$ with the aid of (24) and $\tau'''(0)$, $\tau^{(\mathrm{iv})}(0)$ and $\tau^{(\mathrm{v})}(0)$ by employing (25). Utilizing (26) we can determine τ_1 and τ'_1 and insertion of these last values into (24) and (25) yields τ''_1, τ'''_1, $\tau_1^{(\mathrm{iv})}$ and $\tau_1^{(\mathrm{v})}$. This scheme proceeds in a forward manner until we obtain τ_N and τ'_N.

* It appears at this time that due to the sharp peak of $\tau(x)$ near $x = c/2$ a non-uniform subdivision of the interval may lead to both economy in computations and improved accuracy.

At this stage we can test if the boundary condition (23) is satisfied, namely if

$$\tau'(N) = \frac{P}{2ahE}\frac{1}{R'_N} \tag{27}$$

and adjust the guess $\tau_0^{(1)}$ (and subsequent guesses) to meet this requirement. Unfortunately, this method becomes unstable for sharply peaking values $\tau(c/2)$. We therefore resort to an alternate approach, in which we test for the global equilibrium condition, namely

$$\int_0^{c/2} \tau(x)\,dx = \frac{1}{4}P \tag{28}$$

Consequently, if any guess—say $\tau_0^{(j)}$—resulted in

$$\int_0^{c/2} \tau^{(j)}(x)\,dx = \frac{1}{4}F \neq \frac{1}{4}P$$

we selected the next guess to be

$$\tau_0^{(j+1)} = \frac{P}{F}\tau_0^{(j)} \tag{29}$$

Obviously, (29) represents a linear proportionality and thus may fail to lead to convergence when the non-linearity of our problem asserts itself strongly. Such a circumstance arises when the peak near $x = c/2$ is extremely sharp.*

In our computations the integration (28) was performed by means of Simpson's rule, which was found to provide high accuracy.

The following values were employed in the numerical calculations:

$E = 10^7$ psi (Aluminium)

$D_0 = 0.2 \times 10^{-5}\,(\text{psi})^{-1}$
$D_1 = 0.1 \times 10^{-6}$
$n = 0.2$
$\alpha = 0.125 \times 10^{-3}$
} All typical of epoxy resins[9]

$c = 2''$, $a = 3 \times 10^{-3}{}''$, with $h = 0.25''$ and $1''$.

In all the computation the external load was $P = 5000$ lbs.

The two values of h were employed in order to assess the influence of the joint's stiffness on the stress profile in the adhesive layer.

For the stiffer joint, $h = 1''$, it was sufficient to divide $c/2$ into $N = 100$ subdivisions. However for the more pliant case of $h = 0.25''$ a value of $N = 500$ was required to achieve satisfactory accuracy (accuracy was

* This case, though not encountered in the present computations, may require several modifications, including non-uniform sub-intervals Δ and a non-linear interpolation to replace (29).

checked by comparing results for N subdivisions with values obtained from $2N$ subdivisions).

The computations were performed for times $t = 1, 10, 10^2, \ldots 10^6$ minutes. Results are shown in Figures 3 and 4. In those figures the shear stress τ in the adhesive layer is plotted *vs.* the distance x from the center of each lap to the edge ($0 \leqslant x \leqslant c/2 = 1''$). Note that as time progresses the peak stresses diminish—a phenomenon that is due to the enhances relaxation which occurs in the highly stressed region. A comparison between Figures 3 and 4 shows that the peaks get steeper with decreasing joint stiffness. This observation agrees with conclusions based upon the simpler elastic model.

The elastic solution (13) is not shown in the figures. This solution coalesces with the viscoelastic solution at $t = 0$ and is practically indistinguishable from the curves for $t = 1$.

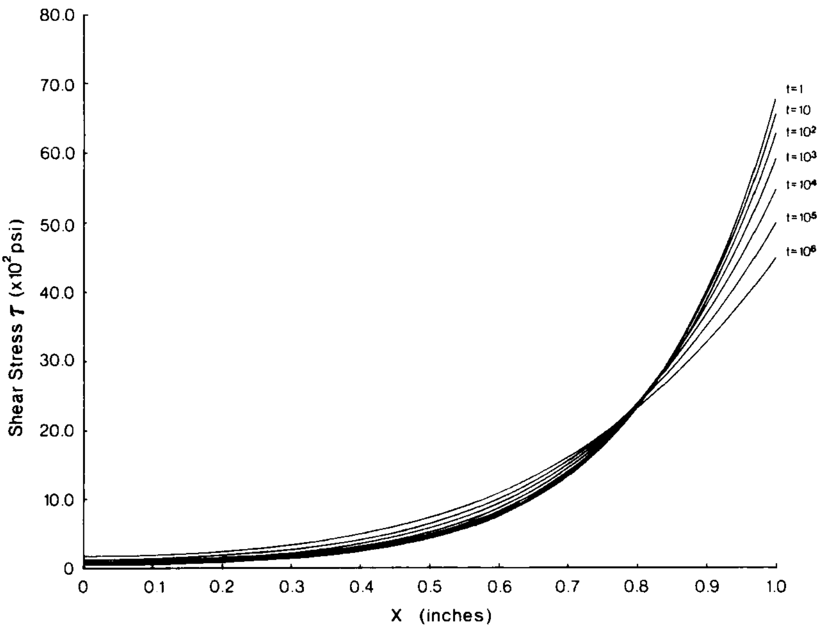

FIGURE 3 Shear stress τ in the bond-Line *vs.* distance x at different times t (t in minutes). Adherent thickness $h = 1$ in.

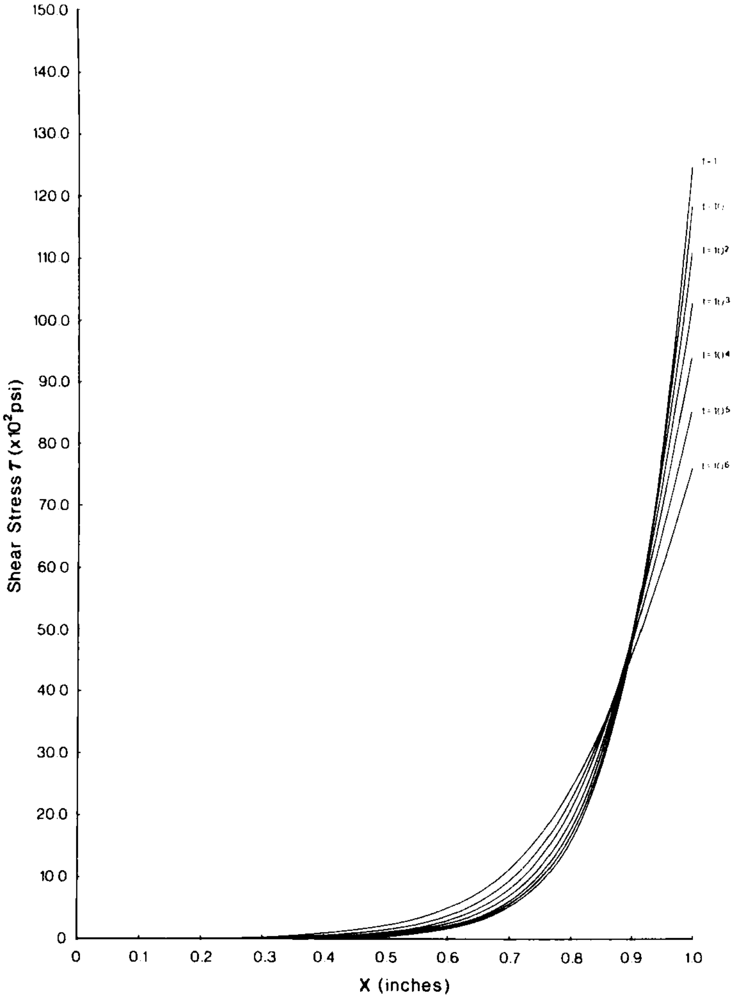

FIGURE 4 Shear stress τ in the bond-Line *vs.* Distance x at different times t (t in minutes). Adherend thickness $h = \frac{1}{4}$ in.

SUMMARY AND CONCLUSIONS

An analysis based upon non-linear viscoelasticity, accounting for a stress enhanced creep response, was provided for the shear stresses along the bond-lines of a symmetric double-lap joint. The computations show that the highly stressed regions are most substantially influenced by viscoelastic creep, which tends to reduce the stress levels near the edges of the adhesive joint.

The accounting for the effects of additional aspects of non-linear visco-elasticity, like modulus softening, is beyond the capability of semi-analytical solutions. Such computations necessitate the employment of finite element methods. Nevertheless the results presented herein should provide insight and a basis for comparison

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